Functietheorie 2Y480. Vraagstukken.

1.1 1 $\frac{1-i}{1+i}=-i$, $\left(\frac{2+i}{3-i}\right)^2=\frac{1}{2}i$, $\frac{3-i}{2+i}=1-i$, $\frac{3+i}{2-i}=1+i$, $\frac{3-i}{2+i}+\frac{3+i}{2-i}=2$

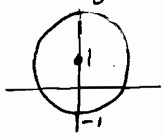
2 a. $|z_1z_2|^2=|(x_1+iy_1)(x_2+iy_2)|^2=|x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)|^2=$
$(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2=x_1^2x_2^2-\cancel{2x_1x_2y_1y_2}+y_1^2y_2^2+x_1^2y_2^2+\cancel{2x_1x_2y_1y_2}$
$+x_2^2y_1^2=(x_1^2+y_1^2)(x_2^2+y_2^2)$

$\arg(z_1z_2)=\arg(z_1)+\arg(z_2)+2k\pi.$
schrijf $x_1=r_1\cos\varphi_1$, $y_1=r_1\sin\varphi_1$, $x_2=r_2\cos\varphi_2$, $y_2=r_2\sin\varphi_2$
$x_1x_2-y_1y_2=r_1r_2(\cos\varphi_1\cos\varphi_2-\sin\varphi_1\sin\varphi_2)=r_1r_2\cos(\varphi_1+\varphi_2)=R\cos(\Phi)$
$x_1y_2+x_2y_1=r_1r_2(\cos\varphi_1\sin\varphi_2+\sin\varphi_1\cos\varphi_2)=r_1r_2\sin(\varphi_1+\varphi_2)=R\sin(\Phi)$

$$\left.\begin{array}{l}\sin(\varphi_1+\varphi_2)=\sin(\Phi)\\ \cos(\varphi_1+\varphi_2)=\cos(\Phi)\end{array}\right\}\to\Phi=\varphi_1+\varphi_2+2k\pi.$$

b. $\bar z_1\bar z_2=(x_1-iy_1)(x_2-iy_2)=x_1x_2-y_1y_2-i(x_1y_2+x_2y_1)=\overline{z_1z_2}$
$1/\bar z=\frac{1}{x-iy}\cdot\frac{x+iy}{x+iy}=\frac{x+iy}{x^2+y^2}$, $\overline{1/z}=\overline{\frac{1}{x+iy}\cdot\frac{x-iy}{x-iy}}=\overline{\frac{x-iy}{x^2+y^2}}=\frac{x+iy}{x^2+y^2}$

c. $(z-i)(\bar z+i)=4.$
$z\bar z+iz-i\bar z+1=x^2+y^2+\cancel{ix}-y-\cancel{ix}-y+1=x^2+y^2-2y+1=$
$x^2+(y-1)^2=4$



cirkel straal 2, mp $=(0,1)$

3 a. $|z_1+z_2|\leq|z_1|+|z_2|.\Leftrightarrow|z_1+z_2|^2\leq(|z_1|+|z_2|)^2\Leftrightarrow$
$(x_1+x_2)^2+(y_1+y_2)^2=x_1^2+2x_1x_2+x_2^2+y_1^2+2y_1y_2+y_2^2\leq|z_1|^2+2|z_1||z_2|+|z_2|^2$
$=x_1^2+y_1^2+x_2^2+y_2^2+2|z_1||z_2|\Leftrightarrow x_1x_2+y_1y_2\leq|z_1||z_2|$
zeker waar als $x_1x_2+y_1y_2<0$; anders :$\Leftrightarrow(x_1x_2)^2+(y_1y_2)^2+2x_1x_2y_1y_2\leq$
$x_1^2x_2^2+x_1^2y_2^2+y_1^2x_2^2+y_1^2y_2^2$

$\Leftrightarrow 2x_1x_2y_1y_2\leq x_1^2y_2^2+y_1^2x_2^2\Leftrightarrow x_1^2y_2^2-2x_1x_2y_1y_2+y_1^2x_2^2=$
$(x_1y_2-y_1x_2)^2\geq0$ QED.

b. $$\left.\begin{array}{l}|z_1|=|z_1+z_2-z_2|\leq|z_1+z_2|+|z_2|\\ |z_2|=|z_2+z_1-z_1|\leq|z_1+z_2|+|z_1|\end{array}\right\}\ \big||z_1|-|z_2|\big|\leq|z_1+z_2|$$

4. a $cz = ax - by + i(ay + bx)$.

$$L_{cz} = \begin{pmatrix} ax-by & -ay-bx \\ ay+bx & ax-by \end{pmatrix}.$$

$$L_c L_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} ax-by, & -ay-bx \\ ay+bx, & ax-by \end{pmatrix} \Big] = .$$

$$L_{c+z} = \begin{pmatrix} a+x & -b-y \\ b+y & a+x \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = L_c + L_z.$$

b. $$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 = 0$$

$$(a-\lambda)^2 = -b^2$$

$$a-\lambda = \pm ib. \rightarrow \lambda = a \pm ib.$$

5 a. $$\left|\frac{z+1}{z}\right| = \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{x^2+y^2}} = 1 \leftrightarrow (x+1)^2 + \cancel{y^2} = x^2 + \cancel{y^2}$$

$$\cancel{x^2} + 2x + 1 = \cancel{x^2} \rightarrow x = -\tfrac{1}{2} \quad \text{(lijn)}.$$

b. $$\left(\frac{z+1}{z}\right)^2 = 1 \leftrightarrow (z+1)^2 = z^2 \quad (z \neq 0)$$

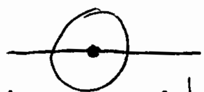
$$z^2 + 2z + 1 = z^2 \leftrightarrow z = -\tfrac{1}{2} \quad \text{(punt)}$$

c. $$z^2 = |z|^2 \leftrightarrow \cancel{x^2} + 2ixy - y^2 = \cancel{x^2} + y^2$$

$$2ixy = 2y^2 \leftrightarrow \underset{\text{reele as}}{y = 0} \vee \underset{\text{(nooit)}}{ix = y}$$

d. $z + \frac{1}{z}$ is reëel? : $x+iy + \dfrac{1}{x+iy} = x+iy + \dfrac{x-iy}{x^2+y^2}$ reëel.

$$y - \frac{y}{x^2+y^2} = 0 \leftrightarrow y = 0 \vee 1 = \frac{1}{x^2+y^2} \leftrightarrow x^2+y^2 = 1.$$



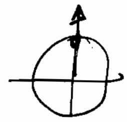
of: $z = re^{i\varphi}$: $\operatorname{Im}\left(re^{i\varphi} + \frac{1}{r}e^{-i\varphi}\right) = r\sin\varphi - \frac{1}{r}\sin\varphi = \left(r - \frac{1}{r}\right)\sin\varphi = 0$

$\rightarrow \varphi = 0 \vee \varphi = \pi$ of $r = 1$.

6. a $\arg\left(\frac{z-i}{z+i}\right)=\frac{\pi}{2}$

$$\frac{z-i}{z+i}=\frac{x+i(y-1)}{x+i(y+1)}\cdot\frac{x-i(y+1)}{x-i(y+1)}=\frac{x^2+(y^2-1)+i(\cancel{xy}-x-\cancel{xy}-x)}{x^2+(y+1)^2}=\frac{x^2+(y^2-1)-2ix}{x^2+(y+1)^2}$$

$$\arg\left(\frac{z-i}{z+i}\right)=\arg\left(x^2+(y^2-1)-2ix\right)=\frac{\pi}{2}.$$

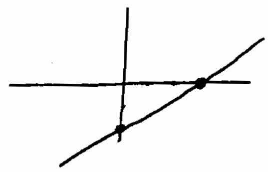


$$\Leftrightarrow \left.\begin{array}{l}-2x\geq 0\\ x^2+y^2-1=0\end{array}\right\},\quad \left.\begin{array}{l}x\leq 0.\\ x^2+y^2=1\end{array}\right\}$$

b. $\mathrm{Im}\left(\frac{z-3}{z+2i}\right)=\frac{x+iy-3}{x+iy+2i}\cdot\frac{x-i(y+2)}{x-i(y+2)}=\frac{(x-3)x+y(y+2)+i(xy-(x-3)(y+2))}{x^2+(y+2)^2}$

$$=\frac{xy-(x-3)(y+2)}{x^2+(y+2)^2}=0\Leftrightarrow \cancel{xy}=\cancel{xy}+2x-3y-6.$$

$$2x-3y=6$$

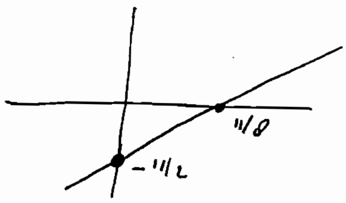


c. $|z-3i|=|4+2i-z|$ (cirkels met gelijke, te bepalen, straal)

$$x^2+(y-3)^2=(4-x)^2+(2-y)^2$$

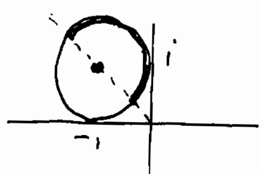
$$\cancel{x^2}+\cancel{y^2}-6y+9=16-8x+\cancel{x^2}+4-4y+\cancel{y^2}$$

$$8x-2y=11$$



d. $|z+1-i|^2=1$? (type fout)

$\frac{\pi}{2}\leq \arg z\leq \frac{3}{4}\pi$



7. a. $z^3=-i \Leftrightarrow r^3e^{3i\theta}=e^{-\frac{1}{2}\pi i+2k\pi i} \to r=1,\ \theta_1=-\frac{1}{6}\pi,\ \theta_2=-\frac{1}{6}\pi+\frac{2}{3}\pi=\frac{\pi}{2}$

$\theta_3=-\frac{1}{6}\pi-\frac{2}{3}\pi=-\frac{5}{6}$

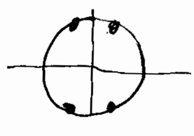
$z_1=\cos\left(\frac{1}{6}\pi\right)-i\sin\left(\frac{1}{6}\pi\right)=\frac{1}{2}\sqrt{3}-\frac{1}{2}i$

$z_2=i$

$z_3=\cos\left(\frac{5}{6}\pi\right)-i\sin\left(\frac{5}{6}\pi\right)=-\frac{1}{2}\sqrt{3}-\frac{1}{2}i$

b. $z^4+2z^2+4=0\Leftrightarrow (z^2+1)^2+3=0\Leftrightarrow (z^2+1)^2=-3,\ z^2+1=\pm\sqrt{3}\,i$

$z^2=-1\pm i\sqrt{3}.=2e^{\pm\frac{2}{3}\pi i}\to z=\pm\sqrt{2}\,e^{\pm\frac{1}{3}\pi i}$ (4 opl.)



7c $z^5 - iz^3 + iz^2 + 1 = 0$

$z^3(z^2 - i) + i(z^2 - i) = 0$

$(z^3 + i)(z^2 - i) = 0 \leftrightarrow z^2 = i \vee z^3 = -i$

$z_1 = i, \quad z_{2,3} = \pm\frac{1}{2}\sqrt{2}(1+i), \quad z_{3,4} = \pm\frac{1}{2}\sqrt{3} - \frac{1}{2}i$

7d. $(z + 2 - i)^6 = 27i = (\sqrt{3})^6 e^{\frac{1}{2}\pi i + 2k\pi i}$

$z_k = -2 + i + \sqrt{3}\exp\left(\frac{1}{12}\pi i + \frac{1}{3}k\pi i\right), \quad k = 0, 1, 2, 3, 4, 5$

8 a. $|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2}) = z_1\bar{z}_1 + z_2\bar{z}_2 - (z_1\bar{z}_2 + \bar{z}_1 z_2)$

$= |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2)$

8 b. zie a: $|z_1 - z_2|^2 + |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$

9. $z\bar{w} = \overline{z\bar{w}} \rightarrow z\bar{w} = \text{reeel} = r, \quad z\bar{w}w = r.w \rightarrow z = cw$

10 a. $|z^2 - c^2| = |z^2|^2 + c^4 - 2\,\mathrm{Re}(z^2c^2) = R^4 + c^4 - 2c^2R^2\cos(2\theta) = 1$

$(R^2 - c^2\cos^2 2\theta)^2 = 1 - c^4 + c^4\cos^2 2\theta = 1 - c^4\sin^2 2\theta.$

$R^2 - c^2\cos^2 2\theta = \pm\sqrt{1 - c^4\sin^2 2\theta}.$

$R^2 = c^2\cos^2 2\theta + \sqrt{1 - c^4\sin^2 2\theta}.$

1.3 1 a. $|z^2 + 2z + 3i| \geq \big||z^2 + 2z| - 3\big| = R|z+2| - 3 \geq R(R-2) - 3$

als $|z| = R > 3$

1 b. $\dfrac{|z-4|}{|z^2 + 2z + 3i|} \leq \dfrac{|R+4|}{R^2 - 2R - 3}$ als $R > 3.$, $\rightarrow 0$ als $R \rightarrow \infty$

Voor $\varepsilon \in (0, \frac{1}{30})$ is $0 < \dfrac{1+4\varepsilon}{1-2\varepsilon-3\varepsilon^2} < 2$, dus als $R = \frac{2}{\varepsilon}$ is $\dfrac{1+4\varepsilon/2}{1-2\frac{\varepsilon}{2}-3\varepsilon^2/4} < \varepsilon.$

2. $\sum_{n=0}^{\infty} r^n \cos n\theta = \sum\left(\frac{1}{2}r^n e^{in\theta} + \frac{1}{2}r^n e^{-in\theta}\right) = \frac{1}{2}\frac{1}{1-re^{i\theta}} + \frac{1}{2}\frac{1}{1-re^{-i\theta}}$

$$= \frac{1-r\cos\theta}{1-2r\cos\theta+r^2}.$$

3. $f(z) = \frac{z^2-1}{|z-1|}$

a. $\lim_{r\to 0} f(1+re^{i\theta}) = \frac{(z-1)(z+1)}{|z-1|} = \frac{re^{i\theta}}{r}(2+re^{i\theta}) = e^{i\theta}(2+re^{i\theta})$

$$\longrightarrow 2e^{i\theta}.$$

b. nee, hangt van θ af.

4. $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a^n b^k\right) = \sum_{n=0}^{\infty} a^n \sum_{k=0}^{n} b^k = \sum_{n=0}^{\infty} a^n \frac{1-b^{n+1}}{1-b} =$

$$= \frac{1}{1-b}\sum_{n=0}^{\infty} a^n - \frac{b}{1-b}\sum_{0}^{\infty}(ab)^n = \frac{1}{1-b}\frac{1}{1-a} - \frac{b}{1-b}\cdot\frac{1}{1-ab} = \frac{a}{(1-a)(1-ab)}$$

convergeert voor $|a|<1$ en $|ab|<1$.

1.4. 1 a. (zie definitie) voor alle $z \neq 1$ is $f(z) = z+1$. Noem $L=2$ terwijl $|z+1-2| = |z-1| < \varepsilon$ mits $0 < |z-1| < \delta = \varepsilon$.

b. $g(z) := f(z)$ $(z\neq 1)$, $g(1) = l = 2$

g is differentieerbaar: bewijs (zie eigenschap 1.4.1)

$$\lim_{h\to 0}\frac{g(1+h)-g(1)}{h} = \frac{1+h+1-2}{h} = \frac{h}{h} = 1 \text{ voor alle } h\neq 0.$$

2. $f(z) = z^n \quad (n \in \mathbb{Z})$.

Bewijs dat f differentieerbaar is, Bepaal afgeleide mbv. definitie.

Zie def. pag 13 of equivalent Eigenschap 1.4.1

$$\lim_{h \to 0} \frac{(z+h)^n - z^n}{h}$$

als $n = 0$: triviaal.

als $n > 0$: $(z+h)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k$.

$$\frac{(z+h)^n - z^n}{h} = \sum_{k=1}^{n} \binom{n}{k} z^{n-k} h^{k-1} \to \binom{n}{1} z^{n-1} = n z^{n-1}$$

als $n = -m < 0$: $\dfrac{(z+h)^{-m} - z^{-m}}{h} = \dfrac{z^m - (z+h)^m}{h z^m (z+h)^m} =$

$$= - \frac{\sum_{k=1}^{m} \binom{m}{k} z^{m-k} h^{k-1}}{z^m (z+h)^m} \to -\binom{m}{1} \frac{z^{-1}}{z^m} = n z^{n-1}.$$

3. $f(z) = z\bar{z}$ is in $z=0$ wel differentieerbaar maar niet holomorf.

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \frac{|h|^2 - 0}{h} = \frac{\rho^2}{\rho e^{i\varphi}} = \rho e^{-i\varphi} \to 0.$$

dus afgeleide is 0.

Beschouw een open omgeving van $z=0$, V.

kies $a \in V$, $a \neq 0$

dan is f niet differentieerbaar

$$\frac{f(a+h) - f(a)}{h} = \frac{|a+h|^2 - |a|^2}{h}.$$

$a = \alpha + i\beta$, $h = \rho e^{i\varphi}$

$$\frac{(\alpha + \rho\cos\varphi)^2 + (\beta + \rho\sin\varphi)^2 - \alpha^2 - \beta^2}{\rho e^{i\varphi}} =$$

$$\frac{2\alpha\rho\cos\varphi + 2\beta\rho\sin\varphi + \rho^2}{\rho e^{i\varphi}} \to 2\,\frac{\alpha\cos\varphi + \beta\sin\varphi}{e^{i\varphi}}$$

hangt af van φ !! Dus niet holomorf.

4. $\cos|z|$ in $z=0$ is wel difbaar maar niet holomorf.

in $z=0$: stel $h = \rho e^{i\varphi}$

$$\frac{\cos(\rho)-1}{\rho e^{i\varphi}} = \frac{1+\frac{1}{2}\rho^2+\dots-1}{\rho e^{i\varphi}} = \frac{1}{2}\rho e^{-i\varphi}+\dots \to 0. \quad (\rho\to 0)$$

in $z\neq 0$: stel $z = r\cos\theta + i r\sin\theta$

$$h = \rho\cos\varphi + i\rho\sin\varphi.$$

dan is $|z+h| = \sqrt{r^2+2r\rho\cos(\theta-\varphi)+\rho^2} = r\sqrt{1+\frac{2\rho}{r}\cos(\theta-\varphi)+\frac{\rho^2}{r^2}}$

$$\simeq r\left(1+\frac{\rho}{r}\cos(\theta-\varphi)+\dots\right) \simeq r+\rho\cos(\theta-\varphi)+\dots$$

$$\cos|z+h| = \cos(r+\rho\cos(\theta-\varphi)+\dots) \simeq \cos r - \rho\sin r\cos(\theta-\varphi)+\dots$$

zodat

$$\frac{\cos|z+h|-\cos|z|}{h} = \frac{\cos r - \rho\sin r\cos(\theta-\varphi)-\cos r}{\rho e^{i\varphi}} =$$

$$\to -\sin r\cos(\theta-\varphi)e^{-i\varphi}$$: hangt af van θ !

5. $f(x+iy) = u(x,y)+i\,v(x,y)$.

CR vgl: $u_x = v_y$, $u_y = -v_x$

$\to$: als $w=f(x+iy)$ heeft: $w_y = i\,w_x$

dan $u_y + i v_y = i u_x - v_x$, dus $u_x = v_y$, $u_y = -v_x$

$\leftarrow$: als $u_x = v_y$, $u_y = -v_x$

dan ook $u_x + i v_x = v_y - i u_y$

dus $i u_x - v_x = i v_y + u_y$, dus $w_y = i w_x$.

6. $p(s,t)$ is polynoom in s en t.

zij $f(z) = p(z,\bar z)$.

f is holomorfe functie desda. p hangt niet van t af.

Bewijs: we hoeven alleen te bewijzen voor een typische term $s^n t^m$

$$(z+h)^n(\bar z+\bar h)^m = z^n\bar z^m + h n z^{n-1}\bar z^m + \bar h m z^n \bar z^{m-1} + O(|h|)$$

zodat $$\frac{(z+h)^n(\bar z+\bar h)^m - z^n\bar z^m}{h} \simeq n z^{n-1}\bar z^m + \frac{\bar h}{h} m z^n \bar z^{m-1}$$

hangt alleen niet af van h als $m=0$: QED

1.5 1. $f(z) = 1/z$

a $\quad \mathrm{Re}(f) = \dfrac{1}{x+iy} \cdot \dfrac{x-iy}{x-iy} = \dfrac{x}{x^2+y^2} = u$

$\mathrm{Im}(f) = \dfrac{-y}{x^2+y^2} = v$

b $\quad$ harmonisch: $u = \dfrac{x}{x^2+y^2}$, $u_x = \dfrac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$

$$u_{xx} = \frac{(x^2+y^2)^2 \cdot -2x - (-x^2+y^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{-2x(-x^2+3y^2)}{(x^2+y^2)^3}$$

$$u_y = \frac{-x}{(x^2+y^2)^2}\cdot 2y$$

$$u_{yy} = -2x\,\frac{x^2-3y^2}{(x^2+y^2)^3}$$

$= 0$

$F = x - u(x^2+y^2)$, $\nabla F = (1-2ux, -2uy)$

$G = +y + v(x^2+y^2)$, $\nabla G = (+2vx, +1+2vy)$

$$\nabla F \cdot \nabla G = (1-2ux)\cdot 2vx - 2uy(1+2vy)$$

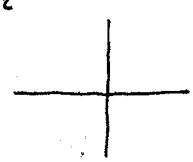
$$= 2vx - 4uvx^2 - 2uy - 4uvy^2$$

$$= 2x\cdot\frac{-y}{x^2+y^2} - 4\cdot\frac{x}{x^2+y^2}\cdot\frac{-y}{x^2+y^2}x^2 - 2\frac{x}{x^2+y^2}y - 4\cdot\frac{x}{x^2+y^2}\cdot\frac{-y}{x^2+y^2}y^2$$

$= 0$

c. $\quad x = u(x^2+y^2)$, $\quad x^2 - \frac{1}{u}x + \frac{1}{4u^2} + y^2 = \frac{1}{4u^2}$

$$\left(x - \frac{1}{2u}\right)^2 + y^2 = \frac{1}{4u^2}$$



$-y = v(x^2+y^2)$, $\quad x^2 + y^2 + \frac{1}{v}y + \frac{1}{4v^2} = \frac{1}{4v^2}$

$$x^2 + \left(y + \frac{1}{2v}\right)^2 = \frac{1}{4v^2}$$

2. a. $\operatorname{Re} f = xy + e^x \cos y$

b. $f(0) = 1+i$

$f(0) = 0 + e^0 \cos 0 + \operatorname{Im} f(0) = 1$

$u = xy + e^x \cos y$, $u_x = y + e^x \cos y = v_y$.

$v = \frac{1}{2}y^2 + e^x \sin y + C(x)$

$f = xy + e^x \cos y + \frac{1}{2} i y^2 + i e^x \sin y + i\, C(x)$.

$= -\frac{1}{2} i (x+iy)^2 + e^x e^{iy} = -\frac{1}{2} i z^2 + e^z + C_0$

$f(0) = 0 + 1 + C_0 = 1+i$

$f(z) = i - \frac{1}{2} i z^2 + e^z$.

3. $\operatorname{Re}(f) = x - \dfrac{x}{x^2+y^2}$, $f(1) = 0$.

$f = z - \dfrac{1}{z} = x + iy - \dfrac{x-iy}{x^2+y^2} = x + iy - \dfrac{x}{x^2+y^2} + i\dfrac{y}{x^2+y^2}$

4. $\operatorname{Re}(f) = e^{ax} \cos(2\pi y)$, $\operatorname{Re} f(1) < 1$, $\operatorname{Im} f(1) = 2\pi$

bepaal a en $f(z)$.

$u_x = a e^{ax} \cos(2\pi y) = v_y \to v = \frac{a}{2\pi} e^{ax} \sin(2\pi y) + C(x)$

$e^{ax} \cos(2\pi y) + i \frac{a}{2\pi} e^{ax} \sin(2\pi y) + i\, C(x)$

$a = 2\pi$: $\quad e^{2\pi x} \cdot e^{2\pi i y} + i\, C$

$\operatorname{Re} f(1) = e^{2\pi} \cos(2\pi) + \cancel{\ldots} = e^{2\pi} \not< 1$

$a = -2\pi$: $\quad e^{-2\pi x} e^{-2\pi i y} + i\, C$.

$\operatorname{Re} f(1) = e^{-2\pi} \cos(2\pi) + \cancel{\ldots} = e^{-2\pi} < 1$ ok.

$\operatorname{Im} f(1) = -e^{-2\pi} \cdot \sin(2\pi) + C = 0 + C = 2\pi$.

dus $f(z) = e^{-2\pi z} + i\, 2\pi$

5. $\mathrm{Re}\, f = x^2y^2$ voor alle $z \in \mathbb{C}$

Kan deze f analytisch (holomorf) zijn?

Kan niet want:

$$u = x^2y^2 \to u_x = 2xy^2 = v_y \to v = \tfrac{2}{3}xy^3 + C(x)$$

$$\to v_x = \tfrac{2}{3}y^3 + C'(x) = -u_y = -2x^2y. \ ↯$$

Beter: $u_{xx} + u_{yy} = 2y^2 + 2x^2 = 2(x^2+y^2) \neq 0.$

6. (a) $\mathrm{Re}(f) = x(x^2 - 3y^2 - 6y - 4) = u.$

$$u_x = 3x^2 - 3y^2 - 6y - 4 = v_y$$

$$\to v = 3x^2y - y^3 - 3y^2 - 4y + C(x)$$

$$v_x = 6xy + C'(x) = -u_y = 6xy + 6x \to C' = 6x,\ C = 3x^2 + C_0$$

$$v = 3x^2y - y^3 - 3y^2 - 4y + 3x^2 + C_0$$

$$f = x^3 - 3xy^2 - 6xy - 4x + i(3x^2y - y^3 - 3y^2 - 4y + 3x^2 + C_0)$$

$$= z^3 + i3z^2 - 4z + iC_0$$

(b) $f(0) = iC_0 = i \to C_0 = 1$

7. $f(z)$ holomorf op V. Laat zien dat $\nabla^2|f|^2 = 4|f'|^2$

$$f = u + iv,\quad |f|^2 = u^2 + v^2.$$

$$\nabla^2(u^2) = \nabla\cdot(2u\nabla u) = 2\nabla u\cdot\nabla u + 2u\nabla^2 u = 2|\nabla u|^2$$

$$\nabla^2 v^2 = 2|\nabla v|^2$$

$$\nabla^2(u^2+v^2) = 2(u_x^2 + v_x^2 + u_y^2 + v_y^2).$$

$$|f'|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 \to 2|f'|^2 = u_x^2 + v_x^2 + u_y^2 + v_y^2$$

$$\to 4|f'|^2 = 2(u_x^2 + v_x^2 + u_y^2 + v_y^2).$$

1.6 1a $\sum_{n=1}^{\infty} \frac{z^{2n}}{2^n}$: $\lim \left|\frac{z^2}{2}\right|^{n+1} / \left|\frac{z^2}{2}\right|^n = \lim \frac{|z|^{2n+2}}{|z|^{2n}} \cdot \frac{2^n}{2^{n+1}} =$

$\lim |z|^2 \cdot \frac{1}{2} = \frac{1}{2}|z|^2 < 1 \rightarrow |z| < \sqrt{2}$.

1b. $\sum_{n=0}^{\infty} (3^n + i^n) z^n$: $\lim \left| \frac{(3^{n+1} + i^{n+1}) z^{n+1}}{(3^n + i^n) z^n} \right| = \frac{3^{n+1}\left(1 + \left(\frac{i}{3}\right)^{n+1}\right)|z|}{3^n\left(1 + \left(\frac{i}{3}\right)^n\right)}$

$\rightarrow 3|z| < 1 \rightarrow |z| < \frac{1}{3}$.

1c. $\sum_{n=0}^{\infty} \frac{n+2}{n!} (z-1)^n \rightarrow \lim_{n\to\infty} \frac{n+3}{\cancel{n!}(n+1)} \cdot \frac{|z-1|^{n+1}}{n+2} \cdot \frac{\cancel{n!}}{|z-1|^n} =$

$\lim_{n\to\infty} \frac{1+3/n}{1+2/n} \cdot \frac{1}{n} \cdot \frac{1}{1+1/n} \cdot |z-1| = \frac{1}{n}|z-1| = 0$

gehele $\mathbb{C}$.

Som: 1a $\sum_{n=1} \frac{z^{2n}}{2^n} = \frac{z^2}{2} \sum_{n=0}^{\infty} \left(\frac{z^2}{2}\right)^n = \frac{1}{2}z^2 \cdot \frac{1}{1-\frac{1}{2}z^2} = \frac{\frac{1}{2}z^2}{1-\frac{1}{2}z^2}$

1b. $\sum_{n=0}^{\infty} (3^n + i^n) z^n = \sum_0 (3z)^n + \sum_0 (iz)^n = \frac{1}{1-3z} + \frac{1}{1-iz}$

1c. $\sum_{n=0} \frac{n+2}{n!}(z-1)^n = \sum_{n=0} \frac{n}{n!}(z-1)^n + 2\sum_0 \frac{1}{n!}(z-1)^n =$

$= \sum_{n=1} \frac{1}{(n-1)!}(z-1)^n + 2e^{z-1} = (z-1) \sum_{n=0} \frac{1}{n!}(z-1)^n + 2e^{z-1} =$

$= ((z-1) + 2) e^{z-1} = (z+1)e^{z-1}$.

2a. $\sum_{n=1} \frac{n!\, z^n}{(1+i)(1+2i)\cdots(1+ni)}$: $\frac{(n+1)!\,|z|^{n+1}}{(1+i)\cdots(1+ni+i)} \cdot \frac{(1+i)\cdots(1+ni)}{n!\,|z|^n} =$

$\left|\frac{n+1}{ni+1+i}\right| \cdot |z| \rightarrow |z| < 1$: $R = 1$

2b: $\sum_{n=1} \frac{z^{n^2}}{(n+1)!}$: $\frac{z^{(n+1)^2}}{(n+2)!} \cdot \frac{(n+1)!}{z^{n^2}} = \frac{1}{n+2} \cdot |z|^{2n+1} < 1$ als $|z| \leq 1$

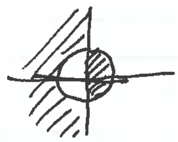
3. $\sum_{n=1} e^{n(z-1/z)} = \frac{e^{z-1/z}}{1-e^{z-1/z}}$ (meetk. reeks).

conv. gebied: als $|e^{z-1/z}|<1$, $\left|e^{x+iy-\frac{x-iy}{x^2+y^2}}\right| = e^{x-\frac{x}{x^2+y^2}} < 1$

$x - \frac{x}{x^2+y^2} < 0.$

$x^2+y^2<1$ als $x>0$.

$x^2+y^2>1$ als $x<0$.



4. $f(z)=\sum_{n=1} \frac{1}{n}\left(\frac{z-1}{z+1}\right)^n$

a. convergentiegebied: $\left|\frac{z-1}{z+1}\right|<1$, $\frac{(x-1)^2+y^2}{(x+1)^2+y^2}<1 \leftrightarrow x>0$

b. $g(w)=\sum_{n=1}\frac{1}{n}w^n$ is holomorf in $|w|<1$ (St. 1,6,5)

$w=\frac{z-1}{z+1}=u+iv$ is holomorf in $x>0$.

dan is $g(u+iv)=p(u,v)+iq(u,v)$ en voldoet aan CR.

$\frac{\partial}{\partial x}\mathrm{Re}\, g = p_u u_x + p_v v_x = +q_v v_y + q_u u_y = \frac{\partial}{\partial y}(\mathrm{Im}\, g)$. etc.

c. $f'(z) = w'\cdot g' = \frac{2}{(z+1)^2}\cdot\sum_{n=0}\left(\frac{z-1}{z+1}\right)^{n} = \frac{2}{(z+1)^2}\cdot\frac{1}{1-\frac{z-1}{z+1}} = \frac{1}{z+1}$

5. $f(z)=\sum_{n=0}\frac{(-1)^n}{2n+1}\cdot\left(\frac{z\sqrt3-1}{z+\sqrt3}\right)^{2n+1}$

a. $\left|\frac{z\sqrt3-1}{z+\sqrt3}\right|<1$ voor convergentie

$2x^2+2y^2-4\sqrt3\, x<2$

$(x-\sqrt3)^2+y^2<4$

$|z-\sqrt3|^2<4$, dus $R=2$

b. zie 4 b

c. $f'(z) = ?$

stel $g(w) = \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n}}{\cancel{}}$, $g(0)=1$

$$\cancel{w} g(w) = \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n}}{\cancel{2n+1}} = \frac{1}{1+w^2} \rightarrow f'(z) = \frac{4}{(z+\sqrt{3})^2} \cdot \frac{1}{1+\left(\frac{z\sqrt{3}-1}{z+\sqrt{3}}\right)^2}$$

$$\cancel{wg'(w) + g(w) = \sum_{n=0}^{\infty} (-1)^n w^{2n} = \frac{1}{1+w^2}} \rightarrow \qquad = \frac{1}{1+z^2}$$

$$\cancel{g(w) = \frac{1}{w} \arctg(w)}.$$

$$\cancel{\text{dus } f'(z) = \frac{d}{dz} g\left(\sqrt{\frac{z\sqrt{3}-1}{z+\sqrt{3}}}\right) = \left(\frac{z+\sqrt{3}}{2\sqrt{3}-1}\right)^{1/2} \arctg\left[\left(\frac{z\sqrt{3}-1}{z+\sqrt{3}}\right)^{1/2}\right]}.$$

~~maar dit is vast de bedoeling niet want complexe wortel en complexe arctg zijn nog niet ingevoerd.~~

6. $f(z) = \sum_{n=0}^{\infty} (n+1)\left(\frac{z}{z-1}\right)^n$.

a. $$\left|\frac{(n+2)\left(\frac{z}{z-1}\right)^{n+1}}{(n+1)\left(\frac{z}{z-1}\right)^n}\right| = \frac{n+2}{n+1} \cdot \left|\frac{z}{z-1}\right| \rightarrow \left|\frac{z}{z-1}\right| < 1.$$

$$x^2+y^2 < (x-1)^2+y^2 = x^2-2x+1+y^2$$

$$0 < -2x+1 \rightarrow x < 1/2.$$

b. $\cancel{f(z) = \frac{d}{dz} \sum \left(\frac{z}{z-1}\right)^{n+1}}$ $g(w) = \sum_{n=1}^{\infty} (n+1) w^n =$

$$= \frac{d}{dw} \sum_{n=1}^{\infty} w^{n+1} = \frac{d}{dw} w^2 \sum_{n=0} w^n = \frac{d}{dw}\left(\frac{w^2}{1-w}\right) = \frac{(1-w)2w+w^2}{(1-w)^2}$$

$$= \frac{2w-2w^2+w^2}{(1-w)^2} = \frac{2w-w^2}{(1-w)^2} = w\,\frac{2-w}{(1-w)^2}$$

$$f(z) = g\left(\frac{z}{z-1}\right) = \frac{z}{z-1} \cdot \frac{2-\frac{z}{z-1}}{\left(1-\frac{z}{z-1}\right)^2} = z \cdot \frac{2(z-1)-z}{(z-1-z)^2} = z(z-2)$$

7. a $\cos^2 z + \sin^2 z = 1$

$$\cos(x+iy) = \frac{1}{2}\left(e^{ix}e^{iiy} + e^{-ix}\cdot e^{-iiy}\right).$$

$$\cos(x+iy)^2 = \frac{1}{4}\left(e^{2ix}e^{2iiy} + 2 + e^{-2ix}\cdot e^{-2iiy}\right).$$

$$\sin(x+iy) = \frac{1}{2i}\left(e^{ix}e^{iiy} - e^{-ix}\cdot e^{-iiy}\right)$$

$$\sin(x+iy)^2 = -\frac{1}{4}\left(e^{2ix}e^{2iiy} - 2 + e^{-2ix}\cdot e^{-2iiy}\right)$$

$$\cos z^2 + \sin z^2 = 0 + \frac{1}{2} + \frac{1}{2} + 0 = 1.$$

b. $\sin(z+w) =$

c. $\cosh(z-w)$

e. $|\sin z|^2 = \cancel{\frac{1}{4}\left(e^{ix}e^{iy} - e^{-ix}e^{+iy}\right)\left(e^{ix}e^{-iy} - e^{-x}e^{iy}\right)}$

$= \frac{1}{4}\cancel{\left(e^{2x} - e^{2iy} - e^{-2iy} + e^{-2x}\right)}$

$= \frac{1}{2}\cancel{\cosh(2x)} - \frac{1}{2}\cos(2y)$

$= \frac{1}{2}ch^2 x + \frac{1}{2}sh^2 x \cancel{+ \frac{1}{2}\cos^2 y + \frac{1}{2}\sin^2 y}$

$= \frac{1}{2}\cancel{(1+2sh^2 x)} = |\sin^2 x\, ch^2 y + \cos^2 x\, sh^2 y| =$

$\sin^2 x(1+sh^2 y) + (1-\sin^2 x)sh^2 y = \sin^2 x + sh^2 y$

d. alle nulpunten van $\sin z$

9. a $e^z = 1+i = \sqrt{2}\, e^{\frac{1}{4}\pi i} = e^x \cdot e^{iy}$.

$$x = \frac{1}{2}\ln 2, \quad y = \frac{1}{4}\pi + 2k\pi.$$

b. $|e^{iz}| = |e^{ix-y}| = e^{-y} = 1 \to y = 0$

c. $\cos z = 10 \to \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y = 10$

$y \neq 0$: $\sin x = 0 \to x = k\pi$, $(-1)^k ch\, y = 10 \to k$ is even

$y = \operatorname{arccosh}(10)$

$x = 2k\pi$

d. $\sin z = 10$.

$\sin x\, ch\, y + i\cos x\, sh\, y = 10 \to \cos x = 0 \to x = \frac{\pi}{2} + k\pi$.

$(-1)^k ch\, y \neq 0 \to k$ is even, $y = \operatorname{arccosh} y$.

$x = \frac{1}{2}\pi + 2k\pi$

10. $\tan z = \frac{1}{i}\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = a.$, $e^{iz}-e^{-iz} = iae^{iz}+iae^{-iz}$

$e^{2iz}-1 = iae^{2iz}+ia.$

$e^{2iz}(1-ia) = 1+ia.$

$e^{2iz} = \frac{1+ia}{1-ia}.$ → ~~alles behalve 1+~~

$= \frac{a-i}{-a-i} = -\frac{a-i}{a+i}$ → alles behalve $a=i$ $(e^z \neq 0)$

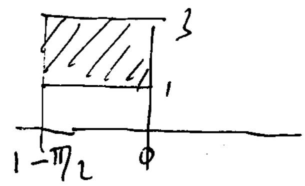
$a=-i$ $(e^z \neq \infty)$

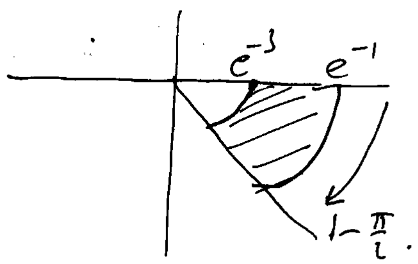
11, (a) $\frac{z^m-1}{z-1}\ \infty \rightarrow z^m = e^{2k\pi i}$ $(\neq 1)$, $z = e^{\frac{2k\pi i}{m}}$, $k=1,2,\dots,m-1$

(b) $\frac{z^m-1}{z-1} = z^{m-1}+z^{m-2}+\dots+z+1 = \prod_{k=1}^{m-1}\left(z-e^{2\pi i\frac{k}{m}}\right).$

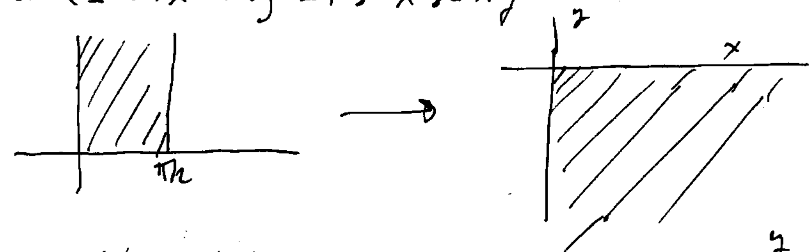
$z=1$: $\prod_{k=1}^{m-1}\left(z-e^{2\pi i\frac{k}{m}}\right) = 1+1+1+\dots+1 = m$

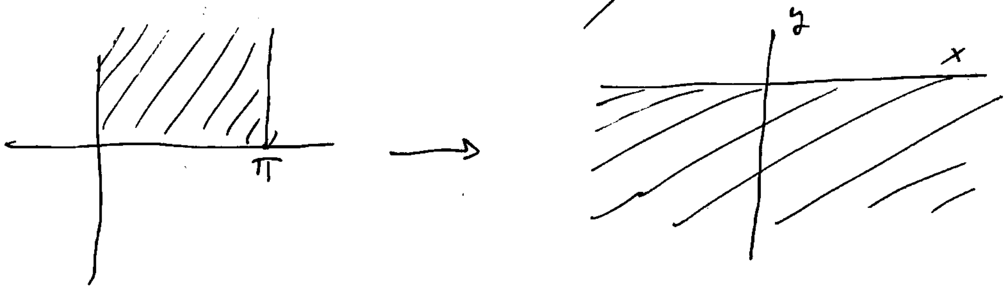
12 e^{ix-y}





13. $\cos z = \cos x\cosh y - i\sin x\sinh y$





Extra opgave Ahlfors: (pag. 37).

(2). Als $\lim_{n\to\infty} z_n = A$, bewijs dat $\lim_{n\to\infty} \frac{1}{n}(z_1+z_2+\cdots z_n) = A$

Bewijs: schrijf $z_n = A + a_n$, $\lim_{n\to\infty} a_n = 0$.

te bewijzen: $\lim_{n\to\infty} \frac{1}{n}(a_1+a_2+\cdots+a_n) = 0$.

Zij gegeven $\varepsilon > 0$..

We weten dat voor een K geldt dat voor $n \geq K$ is $|a_n| < \frac{1}{2}\varepsilon$.

Stel $\max_{n=1..K} |a_n| = M$.

Kies $N \geq \frac{2KM}{\varepsilon}$.

dan is $$\underbrace{a_1+\cdots+a_K}_{\leq M} + \underbrace{a_{K+1}+\cdots+a_N}_{\leq \varepsilon/2}$$

$$\frac{1}{N}(a_1+\cdots+a_N) = \frac{a_1+\cdots a_K}{N} + \frac{a_{K+1}+\cdots a_N}{N}$$

$$\leq \frac{K.M}{N} + \frac{N.\varepsilon/2}{N} \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

(5) Bespreek uniforme convergentie van

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}.$$

merk op: de reeks $\sum \frac{1}{n(1+nx^2)}$ is niet uniform convergent (probleem $x \downarrow 0$)

maar met de x is:

$$\left|\frac{x}{1+nx^2}\right| \leq \frac{1/\sqrt{n}}{1+1} = \frac{1}{2\sqrt{n}} \quad (\text{max in } x = \tfrac{1}{\sqrt{n}})$$

zodat $\left|\frac{x}{n(1+nx^2)}\right| \leq \frac{1}{2n^{3/2}}$ voor alle x: convergent.